xMath 1B - §5.7 – 5.10 Test – Fall 2008 Name______________________________

Instructions: Show your work for credit. Write all responses on separate paper.

1. Consider approximating sums for $\int_1^3 \sin\left(\pi t^2\right) dt$ with $n = 50$ subintervals of equal length.

   a. Write expressions for Δt, t_i., $t_{i\text{-}1}$ and $\dfrac{t_{i-1}+t_i}{2}$.

   b. Approximate to 6 significant digits the midpoint sum M_{50} and its error E_M. Describe what calculator you are using and what buttons you push.

   c. Approximate to 6 significant digits the trapezoidal sum, T_{50}, and its error E_T. Again, describe the buttons you push.

   d. Approximate to 6 significant digits the Simpson sum, S_{100}, and its error E_S.

2. Consider the indefinite integral $\displaystyle\int \frac{x^2+3x+3}{\left(x^2+2x+2\right)\left(x+1\right)}dx$

   a. Find the partial fraction expansion of the integrand. Note that x^2+2x+2 is irreducible, so one term in the expansion is $\dfrac{Ax+B}{x^2+2x+2}$. Find the other term and the values of A, B and C.

   b. Simplify the antiderivative. Note that $x^2+2x+2=\left(x+1\right)^2+1$.

3. Use the substitution $x = 3\tan\theta$ to evaluate $\displaystyle\int \frac{1}{x^2\sqrt{x^2+9}}dx$.

4. Use long division to evaluate the integral $\displaystyle\int \frac{x^4+6}{x^2+9}dx$

5. Use proper limit notation, etc, to show that if $p = 3$ then

   a. $\displaystyle\int_0^1 \frac{dx}{x^{1/p}} = \frac{3}{2}$

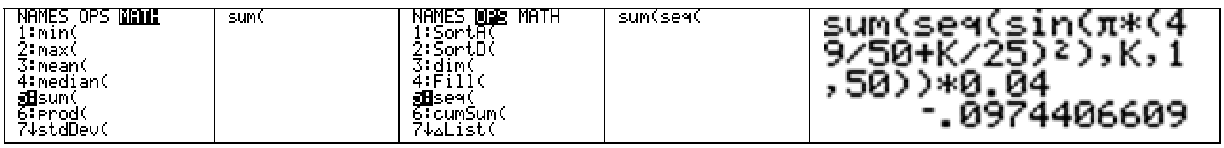
   b. $\displaystyle\int_1^\infty \frac{dx}{x^p} = \frac{1}{2}$

6. Evaluate using integration by parts and proper limit notation, etc: $\displaystyle\int_0^\infty x^2e^{-x}dx$.

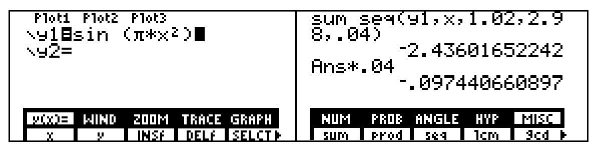
1. Consider approximating sums for $\int_1^3 \sin\left(\pi t^2\right)dt$ with n = 50 subintervals of equal length. Write expressions for Δt, t_i., $t_{i\text{-}1}$ and $\frac{t_{i-1}+t_i}{2}$.

SOLN: $\Delta t = \frac{3-1}{50} = \frac{1}{25} = 0.04$, $t_i = 1+\frac{i}{25}$, $t_{i-1} = 1+\frac{i-1}{25} = \frac{24}{25}+\frac{i}{25}$, $\frac{t_{i-1}+t_i}{2} = \frac{49}{50}+\frac{i}{25}$

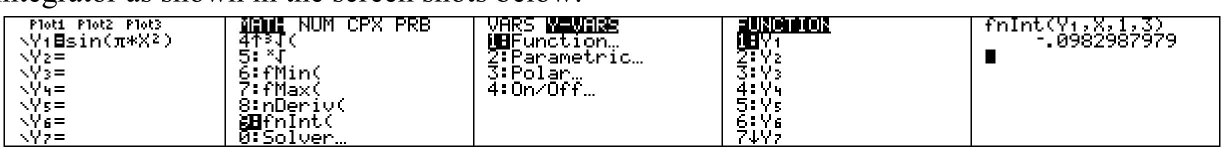
a. Approximate to 6 significant digits the midpoint sum M_{50} and its error E_M. Describe what calculator you are using and what buttons you push.
SOLN: On the TI83 I press the "ON" button (:-D), then 2nd+STAT (LIST) then arrow twice to the right and press "5" to activate "`sum(`". Similarly, get "`seq(`" from the OPS menu under LIST and then enter the formula for evaluating the integrand at midpoints: $\sin\left(\pi\left(1+\frac{2\cdot i-1}{50}\right)^2\right)$ whence I see the screens below (I've swapped i for k and used $\frac{49}{50}+\frac{k}{25} = 1+\frac{2\cdot k-1}{50}$:



Thus the approximation to six significant digits is `-0.0974407`
Alternatively, you can put the function in Y1 and sum Y1 as x goes from 1.02 to 2.98 by steps of 0.04. This is perhaps somewhat simpler and the screenshots for the '86 are shown:



Now we can get an approximation accurate to 12-digits (presumably) on the TI83 by pressing the "MATH" button and then scrolling down to "`fnInt(`" and accessing the built-in numerical integrator as shown in the screen shots below:



Thus the midpoint error with n = 50 is about 0.000858 = $8.58\text{x}10^{-4}$.

b. Approximate to 6 significant digits the trapezoidal sum, T_{50}, and its error E_T. Again, describe the buttons you push.
SOLN: This particular instance of trapezoid sums is simplified by the observation that both the first and the last function value is zero: $f(1) = f(3) = \sin\left(\pi\cdot 1^2\right) = \sin\left(\pi\cdot 3^2\right) = 0$ Thus we can compute the trapezoidal approximation to the area under the curve as the simple average of function values on the interior of the partition times the length of the interval:
$(3-1)\frac{\sum_{i=1}^{49}\sin\left(\pi\left(1+\frac{i}{25}\right)^2\right)}{50} = \frac{1}{25}\left(\sin\frac{26\pi}{25}+\sin\frac{27\pi}{25}+\cdots+\sin\frac{74\pi}{25}\right)$. So on the TI83 we have

Whence the trapezoidal error is -0.0999975693+0.0982987979 = –0.00169877
Note that this is about twice the magnitude of the midpoint error and has the opposite sign.

d. Approximate to 6 significant digits the Simpson sum, S_{100}, and its error E_S.
SOLN: The Simpson sum is

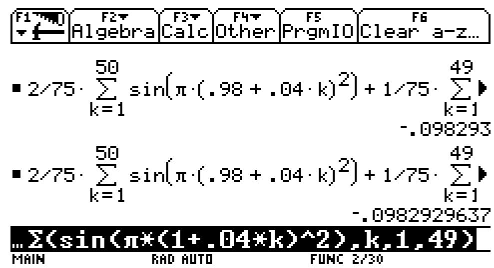
$$\frac{1}{3}\frac{1}{50}\left(f(1)+4f(1.02)+2f(1.04)+4f(1.06)+\cdots+4f(2.98)+f(3)\right)$$ Again,

$f(1)=f(3)=0$, so we can write

$$S_{100}=\frac{4}{150}\left(f(1.02)+f(1.06)+\cdots+f(2.98)\right)+\frac{2}{150}\left(f(1.04)+f(1.08)+\cdots+f(2.96)\right)$$

$$=\frac{2}{75}\sum_{k=1}^{50}\sin\left(\pi\left(\frac{49}{50}+\frac{k}{25}\right)^2\right)+\frac{1}{75}\sum_{k=1}^{49}\sin\left(\pi\left(1+\frac{k}{25}\right)^2\right)$$

On a TI92, you could set this up as shown in this screen shots:



Evidently, $S_{100}\approx-0.0982930$
This, of course, can be more easily computed by taking the weighted average of the trapezoid and midpoint sums:

$$S_{100}=\frac{2M_{50}+T_{50}}{3}\approx\frac{-0.1948813-0.0999976}{3}$$
$$=-0.0982930$$

2. Consider the indefinite integral $\int\frac{x^2+3x+3}{\left(x^2+2x+2\right)(x+1)}dx$

a. Find the partial fraction expansion of the integrand. Note that x^2+2x+2 is irreducible, so one term in the expansion is $\frac{Ax+B}{x^2+2x+2}$. Find the other term and the values of A, B and C.
SOLN: Write the partial fractions form, then clear fractions:

$$\frac{x^2+3x+3}{\left(x^2+2x+2\right)(x+1)}=\frac{Ax+B}{x^2+2x+2}+\frac{C}{x+1}$$

$$x^2+3x+3=(Ax+B)(x+1)+C\left(x^2+2x+2\right)$$

Since this equation must be true for all x, it must be true if $x=-1$ whence $C=1$
Now if $x=0$, we have $3=B+2$, so $B=1$. This makes the equation $x^2+3x+3=(Ax+1)(x+1)+\left(x^2+2x+2\right)=(A+1)x^2+(A+3)x+3$ which indicates that $A=0$. Thus $\int\frac{x^2+3x+3}{\left(x^2+2x+2\right)(x+1)}dx=\int\frac{1}{x^2+2x+2}+\frac{1}{x+1}dx$

b. Simplify the antiderivative. Note that $x^2+2x+2=(x+1)^2+1$.

SOLN: $\int \frac{1}{x^2+2x+2}+\frac{1}{x+1}dx=\int \frac{1}{(x+1)^2+1}+\frac{1}{x+1}dx=\arctan(x+1)+\ln|x+1|+c$.

3. Use the substitution $x=3\tan\theta$ to evaluate $\int \frac{1}{x^2\sqrt{x^2+9}}dx$.

SOLN: $x=3\tan\theta \Rightarrow dx=3\sec^2\theta d\theta$ and $\sqrt{x^2+9}=\sqrt{9\tan^2\theta+9}=3\sec\theta$ whence

$$\int \frac{1}{x^2\sqrt{x^2+9}}dx=\int \frac{1}{9\tan^2\theta(3\sec\theta)}3\sec^2\theta d\theta=\int \frac{\sec\theta}{9\tan^2\theta}d\theta=\frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}d\theta$$

Substituting $u=\sin\theta \Rightarrow du=\cos\theta d\theta$ we have $\frac{1}{9}\int \frac{du}{u^2}=-\frac{1}{9u}+c=-\frac{1}{9\sin\theta}+c$

Since $\theta=\arctan\left(\frac{x}{3}\right)$, $\sin\left(\arctan\left(\frac{x}{3}\right)\right)=\frac{x}{\sqrt{x^2+9}}$ and so $\int \frac{1}{x^2\sqrt{x^2+9}}dx=-\frac{\sqrt{x^2+9}}{9x}+c$

4. Use long division to evaluate the integral $\int \frac{x^4+6}{x^2+9}dx$

SOLN: $\int \frac{x^4+6}{x^2+9}dx=\int x^2-9+\frac{87}{x^2+9}dx=\frac{x^3}{3}-9x+87\int \frac{1}{x^2+9}dx$. Substitute $u = x/3$ to get

$$\int \frac{x^4+6}{x^2+9}dx=\frac{x^3}{3}-9x+29\int \frac{1}{u^2+1}du=\frac{x^3}{3}-9x+29\arctan\left(\frac{x}{3}\right)+c$$

5. Use proper limit notation, etc, to show that if $p = 3$ then

a. $\int_0^1 \frac{dx}{x^{1/p}}=\lim_{b\to 0^+}\int_b^1 x^{-1/3}dx=\lim_{b\to 0^+}\frac{3}{2}x^{2/3}\Big|_b^1=\lim_{b\to 0^+}\frac{3}{2}\left(1-b^{2/3}\right)=\frac{3}{2}$

b. $\int_1^\infty \frac{dx}{x^p}=\frac{1}{2}\int_1^\infty \frac{dx}{x^p}=\lim_{b\to\infty}\int_1^b x^{-3}dx=\lim_{b\to\infty}-\frac{1}{2}x^{-2}\Big|_1^b=\lim_{b\to\infty}\frac{-1}{2}\left(b^{-2}-1\right)=\frac{1}{2}$

6. Evaluate using integration by parts and proper limit notation, etc: $\int_0^\infty x^2e^{-x}dx$.

SOLN:

$$\boxed{\begin{array}{ll} u=x^2 & dv=e^{-x}dx \\ du=2xdx & v=-e^{-x}\end{array}} \mapsto \int_0^\infty x^2e^{-x}dx=\lim_{b\to\infty}\left(-x^2e^{-x}\right)\Big|_0^b+2\int_0^b xe^{-x}dx \mapsto \boxed{\begin{array}{ll} u=x & dv=e^{-x}dx \\ du=dx & v=-e^{-x}\end{array}}$$

$$=\lim_{b\to\infty}-b^2e^{-b}+2\left(-xe^{-x}\right)\Big|_0^b+2\int_0^b e^{-x}dx=\lim_{b\to\infty}\frac{-b^2-2b-2}{e^b}+2=2$$

This last equality follows after two application of L'Hospital's rule